4.4 - Undetermined Coefficients–Superposition Approach

The techniques in this lesson apply to nonhomogeneous DEs of the form $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$ whose input functions (that is, $g(x)$) are constants, polynomials, exponentials, sines and cosines, and sums and products of these.

From 4.1: Consider $y'' - 6y' + 5y = \underbrace{\underbrace{-9e^{2x}}_{g_1(x)} + \underbrace{5x^2 + 3x - 16}_{g_2(x)}}_{g(x)}$

$y_p = Ae^{2x} + Bx^2 + Cx + D$

The complete (general) solution is

$$y = \underbrace{c_1 e^{x} + c_2 e^{5x}}_{y_c \text{ - complementary function}} + \underbrace{\underbrace{3e^{2x}}_{y_{p_1}} + \underbrace{x^2 + 3x}_{y_{p_2}}}_{\text{particular solution } y_p}$$

We can see that $g(x)$ comprises an exponential function and a polynomial.

Starting with $y'' - 6y' + 5y = -9e^{2x} = g_1(x)$

Aux. eqn. $m^2 - 6m + 5 = 0 \Rightarrow (m-5)(m-1) = 0$

$m = 1, 5 \Rightarrow y_c = c_1 e^{x} + c_2 e^{5x}$

Since $g_1(x)$ has the form of an exponential, we "guess" that $y_{p_1} = Ae^{2x} \Rightarrow y_{p_1}' = 2Ae^{2x}$, $y_{p_1}'' = 4Ae^{2x}$

$$\Rightarrow 4Ae^{2x} - 12Ae^{2x} + 5Ae^{2x} = -9e^{2x}$$

$$-3A = -9 \Rightarrow A = 3$$

$$y_{p_1} = 3e^{2x} \Rightarrow y = \underbrace{c_1 e^{x} + c_2 e^{5x}}_{y_c} + \underbrace{3e^{2x}}_{y_{p_1}}$$

quadratic

Consider $g_2(x) = 5x^2 + 3x - 16$ We know $y_c = c_1e^x + c_2e^{5x}$

$y_{p_2} = Ax^2 + Bx + C \Rightarrow y_{p_2}' = 2Ax + B, \; y_{p_2}'' = 2A$

goes into $y'' - 6y' + 5y = 5x^2 + 3x - 16$

$2A - 12Ax - 6B + 5Ax^2 + 5Bx + 5C = 5x^2 + 3x - 16$

x^2 terms: $5A = 5 \Rightarrow A = 1$

x terms: $-12A + 5B = 3 \Rightarrow -12 + 5B = 3 \Rightarrow B = 3$

Constant terms: $2A - 6B + 5C = -16 \Rightarrow -16 + 5C = -16$

$C = 0$ so $y_{p_2} = x^2 + 3x$

$y = y_c + y_p = y_c + y_{p_1} + y_{p_2}$

yields $y = c_1e^x + c_2e^{5x} + 3e^{2x} + x^2 + 3x$

Process:
- Solve associated homog. DE — y_c
- "guess" the form of y_p
- Avoid redundancy using powers of x.
- Sub y_p into nonhomog. DE to find A, B, C, etc.
- Form the solution: $y = y_c + y_p$

Example: Solve the given differential equation by undetermined coefficients.

$$y'' - 8y' + 20y = \underbrace{100x^2}_{g_1(x)} - \underbrace{26xe^x}_{g_2(x)}$$

1st solve homog. DE : $m^2 - 8m + 20 = 0$

$$m^2 - 8m + 16 = -20 + 16$$

$$(m-4)^2 = -4 \Rightarrow m = 4 \pm 2i$$

$$y_c = e^{4x}(c_1 \cos 2x + c_2 \sin 2x)$$

$y_{p_1} = Ax^2 + Bx + C \Rightarrow$ we find

$$y_{p_1} = 5x^2 + 4x + \frac{11}{10}$$

y_{p_2} : product of linear and exp.

$y_{p_2} = \underbrace{(Ax+B)}_{\text{linear}} \underbrace{e^x}_{\text{exp}}$ because $g_2(x) = -26xe^x$

$$y_{p_2}' = (A + Ax + B)e^x \Rightarrow y_{p_2}'' = (2A + Ax + B)e^x$$

$$(2A + Ax + B)\cancel{e^x} - 8(A + Ax + B)\cancel{e^x} + 20(Ax + B)\cancel{e^x} = -26x\cancel{e^x}$$

Constants: $2A + B - 8A - 8B + 20B = 0$

x-terms: $A - 8A + 20A = -26 \Rightarrow A = -2$

Then $13B = -12 \Rightarrow B = -\frac{12}{13}$

$$y = e^{4x}(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right)e^x$$

Example: $y'' - 16y = 2e^{4x}$

Setting ourselvelves up for possible failure:

$y_p = Ae^{4x} \Rightarrow y_p' = 4Ae^{4x}, \quad y_p'' = 16Ae^{4x}$

$16Ae^{4x} - 16Ae^{4x} \neq 2e^{4x}$ contradiction : $\lightning$

Avoiding said failure :

$y_c : \; m^2 - 16 = 0 \Rightarrow m = \pm 4$

$y_c = c_1 e^{4x} + c_2 e^{-4x}$

$y_p = Ae^{4x}$ is redundant.

To avoid the redundancy, use $y_p = Axe^{4x}$

proceed as before

$y_p' = (A + 4Ax)e^{4x}, \quad y_p'' = (4A + 4A + 16Ax)e^{4x}$

$\underbrace{(8A + 16Ax)e^{4x}}_{y_p''} \underbrace{- 16Axe^{4x}}_{-16y} = 2e^{4x} \Rightarrow A = \frac{1}{4}$

$$y = c_1 e^{4x} + c_2 e^{-4x} + \frac{1}{4} x e^{4x}$$

Example: $4y'' - 4y' - 3y = \cos 2x$

$$4m^2 - 4m - 3 = 0 \Rightarrow (2m+1)(2m-3) = 0$$

$$m = -\frac{1}{2}, \frac{3}{2} \Rightarrow y_c = c_1 e^{-1/2 x} + c_2 e^{3/2 x}$$

$$y_p = A\cos 2x + B\sin 2x$$

will yield $y_p = -\frac{19}{425}\cos 2x - \frac{8}{425}\sin 2x$

$$y_p' = -2A\sin 2x + 2B\cos 2x$$

$$y_p'' = -4A\cos 2x - 4B\sin 2x$$

$$-16A\cos 2x - 16B\sin 2x + 8A\sin 2x - 8B\cos 2x$$

$$-3A\cos 2x - 3B\sin 2x = \cos 2x + 0\sin 2x$$

$\cos 2x$: $-16A - 8B - 3A = 1 \rightarrow -19A - 8B = 1$

$\sin 2x$: $-16B + 8A - 3B = 0 \quad 8A - 19B = 0$

I'd proceed using elimination

y_p forms, assuming no redundancy

$g(x)$	y_p
$3x^2 + 2$	$Ax^2 + Bx + C$
$7e^{4x}$	Ae^{4x}
$8x^2e^{2x}$	$(Ax^2 + Bx + C)\, e^{2x}$
$3 \sin 2x$	$A \sin 2x + B \cos 2x$
$x \cos 3x$	$(Ax + B)\cos 3x + (Cx + D)\sin 3x$

(quad)(exp)